Semantics of First Order Predicate Calculus

A model is a pair $\langle \mathcal{D}, \mathcal{A} \rangle$ where:

- $\mathcal{D}$ is a (possibly infinite) non-empty set of individuals called the *domain*, and
- $\mathcal{A}$ is a function that maps
  - each n-ary function symbol to a function from $\mathcal{D}^n$ to $\mathcal{D}$, and
  - each n-ary predicate symbol to a function from $\mathcal{D}^n$ to $\{\mathit{TRUE}, \mathit{FALSE}\}$.

Notice that, as a special case of these rules, a function symbol of zero arity (i.e., a constant or individual symbol) is mapped to an element of $\mathcal{D}$ and a predicate symbol of arity 0 (i.e., a proposition symbol) is mapped to *TRUE* or *FALSE*.

The truth value of a formula with free variables depends not only on what model we are considering but also on what values we give to the variables. For this purpose we will consider the value of an expression relative to a model and a function from variables to $\mathcal{D}$. Such a function will be called a *value assignment*.

If α is an expression or a symbol of the language, then $[\![\alpha]\!]^{I,g}$ is the semantic value of α relative to model I and value assignment g.

If g is a value assignment then $g[x \mapsto i]$ is the same value assignment with the possible exception that $g[x \mapsto i](x) = i$.

Let $I = \langle \mathcal{D}, \mathcal{A} \rangle$ be a model and g be a value assignment. Then $[\![\alpha]\!]^{I,g}$ is defined as follows:

- If α is a constant, function symbol, or predicate symbol then $[\![\alpha]\!]^{I,g} = \mathcal{A}(\alpha)$.
- If α is a variable then $[\![\alpha]\!]^{I,g} = g(\alpha)$.
- If α is a term $f(t_1, \ldots, t_n)$ then $[\![\alpha]\!]^{I,g} = [\![f]\!]^{I,g}([\![t_1]\!]^{I,g}, \ldots, [\![t_n]\!]^{I,g})$.
- If α is an atomic formula $P(t_1, \ldots, t_n)$ then $[\![\alpha]\!]^{I,g} = [\![P]\!]^{I,g}([\![t_1]\!]^{I,g}, \ldots, [\![t_n]\!]^{I,g})$.
- If α is an atomic formula $(t_1 = t_2)$ then

$$[\![\alpha]\!]^{I,g} = \begin{cases} \mathit{TRUE} & \text{if } [\![t_1]\!]^{I,g} = [\![t_2]\!]^{I,g} \\ \mathit{FALSE} & \text{otherwise.} \end{cases}$$

- If α is $(\neg\beta)$ then

$$[\![\alpha]\!]^{I,g} = \begin{cases} \mathit{TRUE} & \text{if } [\![\beta]\!]^{I,g} = \mathit{FALSE} \\ \mathit{FALSE} & \text{otherwise.} \end{cases}$$

- If α is $(\beta \vee \gamma)$ then

$$[\![\alpha]\!]^{I,g} = \begin{cases} \mathit{TRUE} & \text{if } [\![\beta]\!]^{I,g} = \mathit{TRUE} \text{ or } [\![\gamma]\!]^{I,g} = \mathit{TRUE} \\ \mathit{FALSE} & \text{otherwise.} \end{cases}$$

- If α is $(\beta \wedge \gamma)$ then

$$[\![\alpha]\!]^{I,g} = \begin{cases} \mathit{TRUE} & \text{if } [\![\beta]\!]^{I,g} = \mathit{TRUE} \text{ and } [\![\gamma]\!]^{I,g} = \mathit{TRUE} \\ \mathit{FALSE} & \text{otherwise.} \end{cases}$$

- If α is $(\beta \rightarrow \gamma)$ then

$$[\![\alpha]\!]^{I,g} = \begin{cases} \mathit{TRUE} & \text{if } [\![\beta]\!]^{I,g} = \mathit{FALSE} \text{ or } [\![\gamma]\!]^{I,g} = \mathit{TRUE} \\ \mathit{FALSE} & \text{otherwise.} \end{cases}$$

- If α is $(\beta \leftrightarrow \gamma)$ then

$$[\![\alpha]\!]^{I,g} = \begin{cases} \mathit{TRUE} & \text{if } [\![\beta]\!]^{I,g} = [\![\gamma]\!]^{I,g} \\ \mathit{FALSE} & \text{otherwise.} \end{cases}$$

- If α is $(\forall x \beta)$ then

$$[\![\alpha]\!]^{I,g} = \begin{cases} \mathit{TRUE} & \text{if for all } i \in \mathcal{D},\ [\![\beta]\!]^{I,g[x \mapsto i]} = \mathit{TRUE} \\ \mathit{FALSE} & \text{otherwise.} \end{cases}$$

- If α is $(\exists x \beta)$ then

$$[\![\alpha]\!]^{I,g} = \begin{cases} \mathit{TRUE} & \text{if for some } i \in \mathcal{D},\ [\![\beta]\!]^{I,g[x \mapsto i]} = \mathit{TRUE} \\ \mathit{FALSE} & \text{otherwise.} \end{cases}$$

Notes:

If α is a sentence then $[\![\alpha]\!]^{I,g}$ is the same for every value assignment g. That is, $[\![\alpha]\!]^{I,g}$ is independent of g. Therefore, we can omit the g and write $[\![\alpha]\!]^{I}$.

First Order Predicate Calclus with Equality

In the First Order Predicate Calculus with Equality the equality symbol denotes the same function in every model (which is why it is also considered a logical symbol). In particular, it denotes the function $\lambda xy.(\text{if } x = y \text{ then } \mathit{TRUE} \text{ else } \mathit{FALSE})$. Thus,

$$[\![t_1 = t_2]\!]^{I,g} = \begin{cases} \mathit{TRUE} & \text{if } [\![t_1]\!]^{I,g} = [\![t_2]\!]^{I,g} \\ \mathit{FALSE} & \text{otherwise.} \end{cases}$$